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Comments on BRST quantization of strings

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Abstract: The BRST quantization of strings is revisited and the derivation of the path integral measure for scattering amplitudes is streamlined. Gauge invariances due to zero modes in the ghost sector are taken into account by using the Batalin-Vilkovisky formalism. This involves promoting the moduli of Riemann surfaces to quantum mechanical variables on which BRST transformations act. The familiar ghost and antighost zero mode insertions are recovered upon integrating out auxiliary fields. In contrast to the usual treatment, the gauge-fixed action including all zero mode insertions is BRST invariant. Possible anomalous contributions to BRST Ward identities due to boundaries of moduli space are reproduced in a novel way. Two models are discussed explicitly: bosonic string theory and topological gravity coupled to the topological A-model.

Keywords: BRST Quantization, Topological Strings, Conformal Field Models in String Theory, Bosonic Strings.

Contents

1. Introduction

The BRST quantization of strings is a well studied subject, see [1 – 3] for textbook expositions and reviews. In most works however global issues are either ignored or dealt with afterwards. For instance, a naive application of BRST quantization leads to amplitudes that all vanish due to ghost zero modes. To deal with these zero modes, one inserts a number of ghost fields in the path integral measure. Historically, these insertions were first derived by a careful Fadeev-Popov analysis [4 – 8]. One could then show that the path integral in the presence of insertions is BRST invariant up to total derivatives in moduli space. This may look satisfactory, but one should contrast this situation with the BRST quantization of quantum field theories, where the BRST method leads to an action and measure that are both BRST invariant (in the absence of BRST anomalies).

We will show in this paper that one can incorporate global issues in the BRST quantization, leading to a straightforward derivation of the ghost insertions in the path integral. The main observation is that the existence of ghost zero modes implies that the gauge fixed action has additional gauge invariances, namely it is invariant under a variation of the ghosts proportional to their zero modes. The proper way to quantize the theory in such circumstances is to use the BV or antifield formalism [9, 10]. Essentially one introduces a new set of fields and following a well-established procedure arrives at the gauge fixed action.

In the case at hand, we will find that the new fields include the moduli. Furthermore, upon integrating out a set of auxiliary fields one arrives at the usual ghost insertions in the path integral. In other words, the BRST-BV quantization automatically leads to an integral over the moduli space with the correct measure. The resulting gauge-fixed action (which incorporates all ghost insertions) is by construction BRST invariant, so this treatment is exactly analogous to the QFT treatment. One should note that the BRST-BV transformations differ from the ones appearing in most of the literature in that they also act on moduli (by a shift). The treatment of the moduli as quantum mechanical degrees of freedom on which BRST transformations act appeared before in, for example, [11 − 14]. Our point of view is that the standard rules of quantization require such a treatment and this leads to an automatic and simple derivation of the path integral measure.

The current treatment also simplifies and streamlines the derivation of BRST Ward identities [8, 15, 16]. Since our formulation exactly parallels the BRST quantization of quantum field theories, one may just borrow the derivation of Ward identities in QFT where the identities are simply derived by a change of variables in the path integral that amounts to a BRST shift, as in [13]. In the usual treatment one finds that the BRST trivial states decouple only if there is no contribution from the boundary of moduli space. This is so because (in the usual treatment) the path integral measure is BRST invariant only up to total derivatives. In our case, we find the same result but the derivation is somewhat different. In our discussion, the action and measure are BRST invariant. However, BRST transformations shift the boundaries of the integral over moduli; in the derivation of the Ward identity (which involves shifts of fields by their BRST variation) this leads to potential contributions from boundaries of moduli space (which may be "at infinity").

These contributions from boundaries of the integration domain of some of the "fields" in the path integral is the main difference with usual quantum field theories. While in quantum field theory the only source of BRST anomalies is the jacobian of the BRST transformations, in string theory anomalies may also originate from boundary terms. Actually the discussion of anomalies in string theory usually involves the discussion of the boundary terms rather than the computation of jacobians, see for example [17, 18] for discussions of type-I gauge anomalies.

Our considerations apply in general, but for concreteness we present our discussion by means of two examples: the bosonic string and topological strings. The emphasis in both cases is in new features and the derivation of the measure. The latter model illustrates the issue of boundary contributions to Ward identities: the model exhibits the so-called holomorphic anomaly. We will see that the anomaly implies that the theory is gauge dependent.

This paper is organized as follow. In section 2, we briefly review the basics of the BV formalism. In sections 3 and 4, we discuss bosonic and topological strings, respectively. Section 5 contains our conclusions.

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2. BV quantization

In this section we review the basics of Batalin-Vilkovisky (BV) or antifield quantization [9, 10]. In the BV formalism one constructs a BV action from which both the gauge-fixed action and the BRST transformations can be obtained.

The first step is the introduction of an appropriate number of ghosts: for each local symmetry of the action one introduces a set of ghost fields. Let ϕ^i be the fields that a gauge invariant classical action $\mathcal{S}$ depends on. The gauge transformations are given by

$$\delta\phi^i = R^i_{\alpha_0}\epsilon^{\alpha_0}\,, \tag{2.1}$$

where ϵ^{α_0} is the parameter of the transformation and we use De Witt's condensed notation, i.e. the indices can be discrete or continuous; a repeated continuous index includes an integration. Corresponding to these transformations, one introduces ghost fields $C_0^{\alpha_0}$. If $R^i_{\alpha_0}$ are linearly independent in a neighborhood of a stationary point ϕ^i_0 of the action $\mathcal{S}$, then the theory is "irreducible" and $C_0^{\alpha_0}$ are all the ghosts we need (apart from possible extraghosts in the non-minimal sector, which we will discuss later). This will be the case for the systems discussed in this paper. If $R^i_{\alpha_0}$ is not of maximal rank, i.e. if there are non-zero solutions of $R^i_{\alpha_0}Z^{\alpha_0}_{1\alpha_1}|_{\phi_0} = 0$ ($A|_{\phi_0}$ denotes that A is evaluated at the stationary point ϕ_0 of the action $\mathcal{S}$), the theory is "reducible" and one needs an additional set of ghosts. Specifically, reducibility implies that the action for the ghosts $C_0^{\alpha_0}$ has a new gauge invariance given by $\delta C_0^{\alpha_0} = Z^{\alpha_0}_{1\alpha_1}\epsilon_1^{\alpha_1}$. This leads to the introduction of ghosts-for-ghosts $C_1^{\alpha_1}$. Similarly, if $Z^{\alpha_0}_{1\alpha_1}$ is not of maximal rank the action for the ghosts-for-ghosts will have a gauge invariance and we need a new set of ghosts $C_2^{\alpha_2}$, and so on.

Let ϕ^A denote the collection of the fields ϕ^i and of all ghosts $C_0^{\alpha_0}, C_1^{\alpha_1}, \ldots$ We introduce an antifield ϕ^*_A for each field ϕ^A. The antifield ϕ^*_A has opposite statistics compared to the corresponding field ϕ^A, and its ghost number is

$$\mathrm{gh}(\phi^*_A) = -\mathrm{gh}(\phi^A) - 1\,. \tag{2.2}$$

We then define an odd graded Lie bracket, the antibracket:

$$(A,B) = \frac{\delta^R A}{\delta\phi^A}\frac{\delta^L B}{\delta\phi^*_A} - \frac{\delta^R A}{\delta\phi^*_A}\frac{\delta^L B}{\delta\phi^A}\,, \tag{2.3}$$

where right and left derivatives are defined by $\delta F = (\delta^R F/\delta z)\delta z = \delta z(\delta^L F/\delta z)$. The minimal action $S_{\min}[\phi^A, \phi^*_A]$ is given by the solution to the master equation,

$$(S,S) = 0\,, \tag{2.4}$$

with the boundary condition that $S[\phi^A, 0] = \mathcal{S}[\phi^i]$. If the gauge algebra closes off-shell (as in our examples) S is linear in the antifields.

To gauge fix the theory, we need to introduce additional fields, the so-called non-minimal sector: for each set of ghosts $C_i^{\alpha_i}$ we introduce a set of antighosts $b_i^{\beta_i}$,[1] a set of

[1] In general, the ghost field transforms in the adjoint representation of the gauge algebra and the antighost in the co-adjoint. For finite dimensional Lie algebras corresponding to compact Lie groups the adjoint and co-adjoint representations are the same. In general, however, the indices β_i are different from the indices α_i.

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auxiliary fields $\pi_i^{\beta_i}$ and corresponding antifields. Finally, if the gauge fixing condition is such that the ghost action has a new gauge invariance with the antighosts being the gauge fields, extraghosts, corresponding auxiliary fields and antifields are required. We will not review the general case here; the interested reader may consult the original literature and the reviews. If the system is irreducible, as are the systems discussed in this paper, one only needs antighosts $b_0^{\beta_0}$, auxiliary fields $\pi_0^{\beta_0}$, extraghosts $C_0^{k\prime}$, auxiliary fields $\pi_0^{k\prime}$, and antifields for all these fields.

The minimal solution to the master equation is now extended to incorporate the non-minimal sector,

$$S = S_{\min} + b^{0*}_{\beta_0}\pi_0^{\beta_0} + C^{*}_{0k}{}'\pi_0^{k\prime}\,, \tag{2.5}$$

where we only included a single set of antighosts and extraghosts. Gauge fixing is achieved by first performing a canonical transformation, i.e. a transformation $\phi^A \to \phi^{A\prime}(\phi^B, \phi^*_B)$, $\phi^*_A \to \phi^*_A{}'(\phi^B, \phi^*_B)$ that preserves the antibracket, and then setting the antifields to zero, as we now explain.

Canonical transformations are always generated by a fermionic generator Ψ, the so-called gauge fixing fermion,

$$\phi^{A\prime} = e^{\Psi}\phi^A \equiv \phi^A + (\Psi, \phi^A) + \frac{1}{2}(\Psi, (\Psi, \phi^A)) + \cdots \tag{2.6}$$

and similarly for $\phi^*_A{}'$. We define the BV action

$$S_{BV}[\phi^A, \phi^*_A] = S[\phi^{A\prime}, \phi^*_A{}']\,. \tag{2.7}$$

If, as will be the case in this paper, Ψ depends only on the fields, not on the antifields, we obtain

$$S_{BV}[\phi^A, \phi^*_A] = S\left[\phi^A, \phi^*_A + \frac{\partial\Psi}{\partial\phi^A}\right]. \tag{2.8}$$

This action is invariant under the following BRST transformation acting on both the fields and the antifields,

$$\delta_{BRST}\phi^A = (\phi^A, S_{BV}\Lambda)\,, \qquad \delta_{BRST}\phi^*_A = (\phi^*_A, S_{BV}\Lambda)\,, \tag{2.9}$$

where we introduced a constant anticommuting variable Λ such that δ_{BRST} is a derivation rather than an antiderivation. This transformation is nilpotent off-shell.

The gauge-fixed action is obtained from the BV action by simply setting the antifields to zero,

$$S_{gf}[\phi^A] = S_{BV}[\phi^A, 0]\,. \tag{2.10}$$

This action is invariant under the BRST transformations in (2.9) with the antifields set to zero, $\delta_{BRST}\phi^A = (\phi^A, S_{BV}\Lambda)|_{\phi^*_A=0}$.

We now specialize to irreducible theories, the case of interest in this paper. The gauge fixing fermion is usually taken to be of the form

$$\Psi = b_0^{\beta_0}\chi_{\beta_0}(\phi^i) + b_0^{\beta_0}\sigma_{\beta_0 k'}C_0^{k\prime}\,. \tag{2.11}$$

In the absence of the last term this gauge fixing fermion will lead to a δ-function gauge fixing that sets $\chi_{\beta_0}(\phi^i) = 0$. The term with the extraghost is necessary if the gauge fixing condition leads to a ghost action with a new gauge invariance acting on the antighost, as we now explain. Following the steps we outlined above and ignoring for the moment the last term in Ψ one arrives at the ghost action

$$b_0^{\beta_0} \frac{\partial \chi_{\beta_0}}{\partial \phi^i} R^i_{\alpha_0} C^{\alpha_0} \,. \tag{2.12}$$

If the matrix $A_{\beta_0 \alpha_0} = \partial_{\phi^i} \chi_{\beta_0} R^i_{\alpha_0}$ has a left zero eigenvalues, $\bar{Z}_k^{\beta_0} A_{\beta_0\alpha_0}{=}0$, then the ghost action is invariant under the symmetry $\delta b_0^{\beta_0} = \bar{Z}_k^{\beta_0} \epsilon_0^{k\prime}$. The extraghost $C_0^{k\prime}$ is introduced in order to deal with this gauge invariance, and the last term in (2.11) is the corresponding gauge fixing condition. The matrix $\sigma_{\beta_0 k'}$ is any convenient matrix of maximal rank.

Notice that in many cases studied in the literature the number of right zero eigenvalues of $R^i_{\alpha_0}$ is the same as the number of left zero eigenvalues of $A_{\beta_0\alpha_0} = \partial_{\phi^i} \chi_{\beta_0} R^i_{\alpha_0}$ and as result extra-ghosts and ghosts-for-ghosts appear simultaneously. In general, however, the two need not coincide and this is what happens in the case of interest to us. In such cases one can have extra-ghosts without ghosts-for-ghosts (or vice versa).

3. BRST symmetry of the bosonic string

3.1 Gauge invariant action

We now apply the BV formalism to the sigma model describing bosonic closed string theory. The fields ϕ^i include the worldsheet metric $g_{ab}(\sigma)$ and the scalar fields $X^\mu(\sigma)$ corresponding to spacetime coordinates. If the spacetime is flat, the worldsheet action is [1]

$$\mathcal{S}[g_{ab}, X^\mu] = \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} g^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi} \int_M d^2\sigma g^{1/2} R \,, \tag{3.1}$$

where M is a Riemann surface.

The gauge symmetry of the action (3.1) consists of Weyl rescalings of the worldsheet metric and diffeomorphisms of the worldsheet. The infinitesimal gauge variation is given by

$$\delta g_{ab} = 2\omega g_{ab} + \nabla_a \epsilon_b + \nabla_b \epsilon_a \,, \qquad \delta X^\mu = \epsilon^a \partial_a X^\mu \,, \tag{3.2}$$

where $\omega(\sigma)$ and $\epsilon^a(\sigma)$ parametrize the infinitesimal Weyl transformation and diffeomorphism, respectively (they were denoted ϵ^{α_0} in the previous section).

In fact, we will really be interested in computing correlation functions of vertex operators, corresponding to string scattering amplitudes. We proceed by introducing sources ρ^i with Weyl weight one that couple to the vertex operators V_i, which are scalar functionals with Weyl weight minus one. The worldsheet action is then modified as follows,

$$S_0[g_{ab}, X^\mu, \sigma_i^a; \rho^i] = \mathcal{S} + \sum_{i=1}^n \rho^i V_i(\sigma_i) \,. \tag{3.3}$$

This way, (3.3) is invariant under diffeomorphisms and Weyl transformations if we accompany the usual action of those transformations with an explicit shift of σ_i,

$$\delta\sigma_i^a = -\epsilon^a(\sigma_i)\,, \tag{3.4}$$

and we take the sources to be invariant under diffeomorphisms. Differentiating with respect to the sources leads to an insertion of the vertex operators in the path integral.[2] One of our tasks below will be to show that this insertion is accompanied by either a ghost insertion or an integration over σ_i.

In (3.3) we consider the σ_i^a to be fields on the same footing as the fields g_{ab} and X^μ, except that σ_i^a are constant fields, i.e. they do not depend on the coordinates σ^a. In other words, in the path integral we integrate over $\sigma_i^a, g_{ab}, X^\mu$ (and ghosts, antighosts and auxiliary fields introduced during the gauge fixing procedure, as we explain below). This may seem unusual, but we will see that it leads to an elegant derivation of the gauge-fixed path integral.

The role of the index i of the fields ϕ^i in the previous section (not to be confused with the index i used in the present section) is now played by $((ab),\sigma)$ (the indices and argument of $g_{ab}(\sigma)$), (a,i) (the indices of σ_i^a) and (μ,σ) (the index and argument of $X^\mu(\sigma)$). Similarly, the index α_0 in the previous section is now replaced by σ' (the argument of $\omega(\sigma')$) and (c,σ') (the index and argument of $\epsilon^c(\sigma')$). The matrix R is then given by

$$\begin{aligned}
R^{(ab),\sigma}_{\sigma'} &= 2g_{ab}\delta(\sigma-\sigma')\,, && (3.5)\\
R^{(ab),\sigma}_{c,\sigma'} &= (\nabla_a\delta_b^c + \nabla_b\delta_a^c)\delta(\sigma-\sigma')\,, && (3.6)\\
R^{a,i}_{\sigma'} &= 0\,, && (3.7)\\
R^{a,i}_{c,\sigma'} &= -\delta_c^a\delta(\sigma_i-\sigma')\,, && (3.8)\\
R^{\mu,\sigma}_{\sigma'} &= 0\,, && (3.9)\\
R^{\mu,\sigma}_{c,\sigma'} &= \partial_c X^\mu\delta(\sigma-\sigma')\,. && (3.10)
\end{aligned}$$

3.2 Gauge fixed action

The first step in the BV procedure is to introduce ghost and auxiliary fields. In our case we have fermionic ghost fields $C_\omega(\sigma)$ (for the Weyl transformations) and $c^a(\sigma)$ (for the diffeomorphisms); these ghost fields were denoted $C_0^{\alpha_0}$ in the previous section. We also introduce several pairs of auxiliary fields. A first pair corresponds to $\tilde{b}^{ab}(\sigma)$ (a fermionic antighost) and $\pi^{ab}(\sigma)$ (a boson with ghost number zero). A second pair is formed by constant fermionic antighosts b_a^j and ghost number zero bosons p_j^a for a set of values

$$(a,j)\in f\,, \tag{3.11}$$

where f will correspond to the set of fixed vertex operator coordinates. In particular, $(a,j) = 1,\ldots,\kappa$ and κ is the number of conformal Killing vectors: $\kappa = 6$ for a Riemann surface of genus zero, $\kappa = 2$ for genus one and $\kappa = 0$ for higher genus. A third pair consists

[2] Note that for an n-point function we introduce n sources even when some of the operators are the same and the sources are always treated infinitesimally, i.e. we only differentiate once with respect to each source and then set all sources to zero.

of constant extraghosts τ^k (bosons of ghost number zero) and corresponding fermionic fields ξ^k of ghost number one. Here $k = 1, \ldots, \mu$, with μ the number of metric moduli: $\mu = 0$ for genus zero, $\mu = 2$ for genus one and $\mu = 6g - 6$ for higher genus. In fact, the extraghosts τ^k will be interpreted as metric moduli.

In the antifield formalism, the action takes the form

$$\begin{aligned} S = S_0 + \int d^2\sigma \Big(& g_*^{ab} \left[2C_\omega(\sigma) g_{ab}(\sigma) + \nabla_a c_b(\sigma) + \nabla_b c_a(\sigma)\right] + \tilde{b}^*_{ab}(\sigma)\pi^{ab}(\sigma) - \\ & - c_a^*(\sigma) c^b \partial_b c^a(\sigma) C_w^*(\sigma) c^b \partial_b C_w(\sigma) + - X_\mu^*(\sigma) c^a \partial_a X^\mu \Big) - \\ & - \sum_{i=1}^{n} \sigma_a^{i*} c^a(\sigma_i) + \sum_{(a,j)\in f} b_a^{j*} p_j^a + \sum_{k=1}^{\mu} \tau_k^* \xi^k \,, \end{aligned} \tag{3.12}$$

where the fields in the last line are constant fields. (The summation over the repeated vector indices is implicit throughout this paper.) Notice that the antifields in (3.12) transform as densities. One could have introduced instead explicit factors of $g^{1/2}$, but the present convention simplifies some of the computations below.

We now discuss gauge fixing. Equivalence classes of metrics under diffeomorphisms and Weyl transformations are labelled by coordinates τ^k. We choose a smooth set of reference metrics $\hat{g}_{ab}(\tau^k;\sigma)$ (which can be and are chosen to have constant curvature). Further, we choose a collection $\hat{\sigma}_i^a$ of reference values for those fields σ_i^a with $(a,i) \in f$. We can then gauge fix the action as follows. For the gauge fermion we make the following choice:

$$\Psi = \frac{1}{4\pi} \int d^2\sigma \, \tilde{b}^{ab}(\sigma)(g_{ab}(\sigma) - \hat{g}_{ab}(\tau^k;\sigma)) + \sum_{(a,j)\in f} b_a^j(\sigma_j^a - \hat{\sigma}_j^a) \,. \tag{3.13}$$

The first term imposes the gauge condition

$$\chi_{ab}(\sigma) = g_{ab}(\sigma) - \hat{g}_{ab}(\tau^k;\sigma) = 0 \,. \tag{3.14}$$

Let us now motivate why we introduced the moduli τ^k as (constant) fields in the action. If we had not done so, but merely considered τ^k as parameters instead of fields, the gauge fixing fermion (3.13) would have led (following the steps we discuss below) to the usual ghost action[3]

$$S_{gh} = \frac{1}{2\pi} \int d^2\sigma \tilde{b}^{ab}(P_1 c)_{ab} \,, \tag{3.15}$$

where the operator P_1 maps vectors to symmetric traceless tensors, $(P_1 c)_{ab} = \frac{1}{2}(\nabla_a c_b + \nabla_b c_a - g_{ab}\nabla_c c^c)$. This ghost action has additional invariances because of the zero modes of P_1 and $P_1^\dagger$: the action is invariant under a shift of c^a by a conformal Killing vector as well as under a shift of $\tilde{b}^{ab}$ by a holomorphic quadratic differential. The symmetry due to ghost zero modes is gauge fixed by fixing the positions of κ vertex operators. This is the origin of the last term in (3.13) which enforces the gauge condition,

$$\chi_j^a = \sigma_j^a - \hat{\sigma}_j^a = 0 \,, \qquad (a,j) \in f \,. \tag{3.16}$$

[3]Note that $\tilde{b}^{ab}$ transforms as a density (in addition to the transformation implied by its indices). In the literature one often uses a tensor field b_{ab} related to our $\tilde{b}^{ab}$ by $\tilde{b}^{ab} = \sqrt{g} g^{ac} g^{bd} b_{cd}$.

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In the absence of vertex operators in (3.3), the presence of ghost zero modes would imply that the gauge algebra is reducible and one would proceed by introducing ghosts-for-ghosts, as described in the previous section. The antighost zero modes also lead to an invariance of the action, as discussed below (2.11). To deal with this invariance we interpret the moduli as extraghost fields, playing the role of $C_0^{k'}$ in (2.11). Compared with the previous section, $\partial_k \hat{g}_{ab}(\tau^k;\sigma)$ is what we called $\sigma_{\beta_0 k'}$. Recall that the tangent space to the moduli space at $\hat{g}_{ab}(\tau^k;\sigma)$ (which is a metric of constant curvature) is spanned by the quadratic differentials $\mathrm{Ker}P_1^\dagger$, so $\partial_k \hat{g}_{ab}(\tau^k;\sigma)$ has indeed maximal rank.

The gauge fixing fermion in (3.13) leads to the gauge fixed action

$$\begin{aligned} S_{gf} = S_0 &+ \frac{1}{4\pi}\int d^2\sigma\, \tilde{b}^{ab}(\sigma)\left[2C_\omega(\sigma)g_{ab}(\sigma) + \nabla_a c_b(\sigma) + \nabla_b c_a(\sigma) - \xi^k\partial_k\hat{g}_{ab}(\tau;\sigma)\right] + \\ &+\frac{1}{4\pi}\int d^2\sigma\, \pi^{ab}(\sigma)\left[g_{ab}(\sigma) - \hat{g}_{ab}(t_0;\sigma)\right] + \\ &+ \sum_{(a,j)\in f}\left[-b_a^j c^a(\sigma_j) + p_a^j(\sigma_j^a - \hat{\sigma}_j^a)\right]. \end{aligned} \tag{3.17}$$

This concludes the construction of the gauged fixed action. The gauged fixed action one often finds in the literature does not contain the last line and the last term in the first line. In addition, C_ω and π^{ab} have usually been integrated out.

We thus arrive at the following generating functional of string amplitudes,

$$Z[\rho^i] = \int d\mu e^{-S_{gf}} \tag{3.18}$$

where

$$d\mu = \prod_{i=1}^{n} d^2\sigma_i\sqrt{g(\sigma_i)}\prod_{j=1}^{\kappa}(db^j dp_j)\prod_{k=1}^{\mu}(d\tau^k d\xi^k)[dX^\mu][dg_{ab}][d\pi_{ab}][d\tilde{b}^{ab}][dC_w][dc^a]\,. \tag{3.19}$$

Many of the fields are auxiliary and can be integrated out, as we discuss in subsection 3.4.

3.3 BRST transformations

The action (3.17) is designed to satisfy the master equation and is thus BRST invariant by construction. One can verify this almost by inspection as we now discuss. Recall that the BRST transformation of a field ϕ^A appearing in the action can be read off from the BV action:

$$\delta_{BRST}\phi^A = \frac{\delta^L S}{\delta\phi_A^*}\Lambda\,, \tag{3.20}$$

where Λ is an anticommuting parameter. In particular,

$$\begin{aligned} \delta_{BRST}X &= c^a\partial_a X\Lambda\,, \\ \delta_{BRST}g_{ab} &= (c^c\partial_c g_{ab} + \partial_a c^c g_{cb} + \partial_b c^c g_{ac} + 2C_\omega g_{ab})\Lambda\,, \\ \delta_{BRST}c^b &= -c^a\partial_a c^b\Lambda\,, \\ \delta_{BRST}C_\omega &= -c^a\partial_a C_\omega\Lambda\,, \\ \delta_{BRST}\tau^k &= \xi^k\Lambda\,, \end{aligned}$$

$$\begin{aligned}\delta_{BRST}\tilde{b}^{ab}(\sigma) &= \pi^{ab}(\sigma)\Lambda\,,\\ \delta_{BRST}\sigma_i^a &= -c^a(\sigma_i)\Lambda\,,\\ \delta_{BRST}b_a^j &= p_a^j\Lambda\,,\end{aligned} \tag{3.21}$$

and $\pi^{ab}, \xi^k, p_a^j, c^a(\sigma^i)$ are BRST invariant.[4]

Defining the BRST charge Q_B by

$$\delta_{BRST}\Phi = \{Q_B, \Phi\}\Lambda\,, \tag{3.22}$$

the gauge fixed action (3.17) can be written as

$$S_{gf} = S_0 + \left\{Q_B, \frac{1}{4\pi}\int d^2\sigma\, b^{ab}[g_{ab} - \hat{g}_{ab}(\tau)] + \sum b_a^j(\sigma_j^a - \hat{\sigma}_j^a)\right\}. \tag{3.23}$$

This equation is familiar in BRST quantization: to gauge fix, one adds a BRST exact term to the original action. The reason we went through the more elaborate BV formalism is to motivate the fact that that the moduli τ^k transform under the BRST symmetry. From (3.23) and the nilpotency of δ_{BRST}, it is clear that the full gauge fixed action is invariant under our BRST transformations.

3.4 Integrating out auxiliary fields

To make contact with the textbook expressions for the gauge fixed action and of string amplitudes, we now integrate out a number of auxiliary fields from the action (3.17).

We will integrate out the fields C_ω, $\tilde{b}_a^a = g_{ab}\tilde{b}^{ab}$, b_a^i, π^{ab}, p_a^i, g_{ab}, ξ^k, ξ_i^a and ρ^a. Performing the functional over the fermionic field C_ω produces insertions of the various modes of $\tilde{b}_a^a(\sigma)$ in the path integral. These insertions effectively set $\tilde{b}_a^a$ equal to zero in the action; they disappear upon performing the functional integral over $\tilde{b}_a^a(\sigma)$. As mentioned before, integrating out the bosonic field $\pi^{ab}(\sigma)$ enforces

$$g_{ab} = \hat{g}_{ab}(\tau^k)\,. \tag{3.24}$$

Similarly, integrating out the bosonic fields p_a^i sets

$$\sigma_i^a = \hat{\sigma}_i^a \qquad \text{if } (a,i) \in f\,. \tag{3.25}$$

Finally, integrating out ξ^k, p_i^a and b_a^i and differentiating with respect to ρ_i and setting $\rho_i = 0$, leads to the familiar path integral

$$\begin{aligned}\langle V_1(k_1)\cdots V_n(k_n)\rangle = \sum_{g=0}^{\infty}\int d^\mu\tau \int [dX\, d\tilde{b}\, dc]\exp(-\mathcal{S} - S_{gh}) \times\\ \times \prod_{(a,i)\in f}\sqrt{\hat{g}(\hat{\sigma}_i)}c^a(\hat{\sigma}_i)V_i(k_i,\hat{\sigma}_i)\prod_{k=1}^{\mu}\frac{1}{4\pi}(\tilde{b}, \partial_k\hat{g}) \times\\ \times \prod_{(a,i)\notin f}\int d\sigma_i^a\sqrt{\hat{g}(\sigma_i)}V_i(k_i,\sigma_i)\,,\end{aligned} \tag{3.26}$$

[4]Nilpotency of the BRST transformation requires $\delta_{BRST}c^a(\sigma_i) = 0$, and one can verify that this holds, $\delta_{BRST}c^a(\sigma_i) = -c^b\partial_b c^a(\sigma_i)\Lambda + \partial_b c^a(\sigma_i)[-c^b(\sigma_i)\Lambda] = 0$.

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where $(\tilde{b}, \partial_k \hat{g}) = \int d^2\sigma \, \tilde{b}^{ab} \partial_k \hat{g}_{ab}$. The ghost and antighost insertions in the path integral are explained as follows. Starting from (3.17) and integrating out p_a^i and b_a^i introduces κ delta functions that fix the positions of κ vertex operators and inserts the ghost $c^a(\hat{\sigma}_i)$ in the path integral. This is the familiar ghost insertion that accompanies fixed vertex operators. The antighost insertions come from the integral over ξ^k.

3.5 Do BRST exact states decouple?

Since the full gauge fixed action (3.17) is BRST invariant, one might think that if the path integral measure is invariant, BRST exact states should decouple. The formal argument, familiar from the derivation of BRST Ward identities in gauge theories, goes as follows. Let $V[\phi^A]$ be an arbitrary function of all fields and consider

$$\langle V[\phi^A] \rangle_{\rho^i} \equiv \int d\mu V[\phi^A] e^{-S_{gf}[\phi^A;\rho^i]} , \qquad (3.27)$$

where ϕ^A denotes collectively all fields we integrate over in the path integral (see (3.19)) and $\langle V[\phi^A] \rangle_{\rho^i}$ indicate the 1-point function of $V[\phi^A]$ in the present of sources, i.e. this expression encompasses arbitrary n-point functions of $V[\phi^A]$ with other vertex operators (obtained by differentiating the 1-point function with respect to the sources and then setting the sources to zero). We now change variables

$$\phi^{A\prime} = \phi^A + \delta_{BRST} \phi^A . \qquad (3.28)$$

Provided that the measure is invariant, i.e. that there are no BRST anomalies, one finds

$$\int d\mu' V[\phi^{A\prime}] e^{-S_{gf}[\phi^{A\prime};\rho^i]} = \int d\mu V[\phi^A] e^{-S_{gf}[\phi^A;\rho^i]} + \int d\mu (\delta_{BRST} V[\phi^A]) e^{-S_{gf}[\phi^A;\rho^i]} , \qquad (3.29)$$

which implies

$$\langle \delta_{BRST} V[\phi^A] \rangle_{\rho^i} = 0 . \qquad (3.30)$$

In other words, BRST exact states seem to decouple from the correlation functions of arbitrary number of BRST invariant vertex operators.

However, there is a loophole in this argument. For the moduli fields, the transformation (3.28) reads

$$\tau^{k\prime} = \tau^k + \xi^k \Lambda , \qquad (3.31)$$

that is, the change of variables shifts the moduli. Now if the integral over moduli gets contributions from boundaries of the integration domain (which may be at infinity), i.e. from boundaries of moduli space, the shift gives rise to extra boundary terms on the right hand side of (3.29). So BRST exact states only decouple if the boundary terms vanish.

To see this more explicitly, single out a modulus τ^0 and assume that the moduli space has a boundary at $\tau^0 = \tau^{0(f)}$: $\tau^0 \leq \tau^{0(f)}$. The boundary may be at infinity or at finite value. (One may similarly incorporate a boundary located at the lower end of the integration domain of τ^0.) Let us write the path integral measure as $d\mu = d\tau^0 d\tilde{\mu}$, and similarly $V[\phi^A] = V[\tau^0, \tilde{\phi}^A]$, where a tilde denotes that τ^0 is excluded. Running the

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previous argument, noting that $\delta\tau^0 = \xi^0\Lambda$ and keeping track of contributions from the boundary of the τ^0 integration domain, one finds

$$\begin{aligned}
\int^{\tau^{0(f)}} d\tau^{0\prime} \int d\tilde{\mu}' V[\tau^{0\prime}, \tilde{\phi}^{A\prime}] e^{-S_{gf}[\tau^{0\prime}, \tilde{\phi}^{A\prime};\rho^i]} \\
&= \int^{\tau^{0(f)}-\xi^0\Lambda} d\tau^0 \int d\tilde{\mu} V[\tau^0+\xi^0\Lambda, \tilde{\phi}^A + \delta\tilde{\phi}^A] e^{-S_{gf}[\tau^0+\xi^0\Lambda, \tilde{\phi}^A+\delta\tilde{\phi}^A;\rho^i]} \\
&= \int^{\tau^{0(f)}} d\tau^0 \int d\tilde{\mu} V[\tau^0, \tilde{\phi}^A] e^{-S_{gf}[\tau^0,\tilde{\phi}^A;\rho^i]} + \\
&\quad + \int^{\tau^{0(f)}} d\tau^0 \int d\tilde{\mu} \left(\delta_{BRST} V[\tau^0, \tilde{\phi}^A]\right) e^{-S_{gf}[\tau^0,\tilde{\phi}^A;\rho^i]} - \\
&\quad - \int d\tilde{\mu}\xi^0 \Lambda V[\tau^{0(f)}, \tilde{\phi}^A] e^{-S_{gf}[\tau^{0(f)},\tilde{\phi}^A;\rho^i]}
\end{aligned} \tag{3.32}$$

which implies

$$\langle \delta_{BRST} V[\phi^A] \rangle_{\rho^i} = \langle \partial_{\tau^0} \left(\xi^0 \Lambda V[\phi^A]\right) \rangle_{\rho^i} \,. \tag{3.33}$$

Now because of the ξ^0 insertion in the right hand side of (3.33), the integral over ξ^0 will not bring down the usual $(\tilde{b}, \partial_0 \hat{g})$ antighost insertion: the boundary term has one less antighost insertion compared to the bulk terms.

In the textbook treatment of BRST quantization of strings, these boundary terms arise in a different way. There the BRST transformations do not act on moduli, and it is the gauge fixed action excluding terms giving rise to antighost insertions that is BRST invariant. The antighost insertions themselves are not invariant because the antighosts transform into the stress tensor. To see this, use (3.21), combined with the equation of motion for g_{ab}:

$$\pi^{ab} = -4\pi \frac{\delta S_2}{\delta g_{ab}} = g^{1/2}\, \Theta^{ab} \,, \tag{3.34}$$

where Θ^{ab} is the stress tensor of the action (3.17) with the term involving π^{ab} omitted. The resulting stress tensor insertion in turn gives rise to a total derivative on moduli space, which upon integration over moduli space leads to boundary terms.

3.6 Summary

In this section, we have used the Batalin-Vilkovisky formalism to derive a manifestly BRST invariant action for the bosonic string. This action includes terms that give rise to ghost and antighost insertions upon integrating out auxiliary fields. In the path integral, the integrals over moduli and vertex operator positions are automatically present, because moduli and vertex operator positions are considered to be (constant) fields in the action. A notable feature of this formalism is that BRST transformations act on moduli; this leads to potential non-decoupling of BRST exact states due to contributions from boundaries of moduli space.

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4. Topological strings

In this section we consider topological gravity coupled to the topological sigma A-model. This model exhibits the so-called holomorphic anomaly [19]: certain BRST exact terms do not decouple because of contributions from boundary terms. One of the motivations for this work was to understand the implications of the anomaly. Usually breaking of BRST invariance in QFT implies lack of renormalizability and unitarity, but these do not seem to be an issue for topological theories. Another implication of the non-decoupling of BRST exact states is that the quantum theory is gauge dependent: shifting the gauge fixing term by the BRST exact term corresponding to the state that does not decouple leads to an inequivalent theory. Thus, the holomorphic anomaly implies that topological string theory is gauge dependent. It is unclear to us what is the proper worldsheet interpretation of this fact, but we note that the holomorphic anomaly has also been linked to a quantum version of background independence [20] (see also [21]).

4.1 The topological sigma model

In this subsection, we briefly review the A-model topological sigma model [22] as constructed in [23]. We restrict our attention to target spaces that are Calabi-Yau manifolds.

The starting point is the action

$$I[X] = \int_M (\omega_{\mu\nu} + iB_{\mu\nu}) dX^\mu \wedge dX^\nu = \int_M dz d\bar{z} (\omega_{\mu\nu} + iB_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu \,, \tag{4.1}$$

where the worldsheet M is a Riemann surface, ω the Kähler form of the Kähler metric $G_{\mu\nu}$ on the target space N,

$$\omega_{i\bar{j}} = -iG_{i\bar{j}} \,, \qquad \omega_{\bar{j}i} = iG_{i\bar{j}} \,, \tag{4.2}$$

B an antisymmetric tensor potential with zero field strength, and X a smooth map from M to N. (The B-field was not present in [23], but will play a role when we discuss the holomorphic anomaly. Its inclusion complexifies the space of Kähler deformations of N.) The action is independent of the worldsheet metric and only depends on the cohomology class of the Kähler form and the homotopy class of the map X. As a consequence, it has a gauge symmetry corresponding to arbitrary small deformations of X:

$$\delta X^\mu = \epsilon^\mu \,. \tag{4.3}$$

This gives rise to the BRST symmetry

$$\begin{aligned} \delta_S X^\mu &= \psi^\mu \,, \\ \delta_S \psi^\mu &= 0 \,, \\ \delta_S \bar{\psi}^\mu &= b^\mu \,, \\ \delta_S b^\mu &= 0 \,, \end{aligned} \tag{4.4}$$

where ψ^μ is a ghost field, $\bar{\psi}^\mu$ an antighost field and b^μ an auxiliary field.

In terms of the complex structure J of the target space (given by $J^\mu{}_\nu = G^{\mu\rho}\omega_{\rho\nu}$), define

$$\dot{X} = (1 - iJ)\bar{\partial}X + (1 + iJ)\partial X\,, \tag{4.5}$$

or in other words

$$\dot{X}^i = 2\bar{\partial}X^i\,, \qquad \dot{X}^{\bar{j}} = 2\partial X^{\bar{j}}\,. \tag{4.6}$$

In [23] the following gauge fixing was chosen:

$$\begin{aligned} I_{gf} = & \int dz d\bar{z}(\omega_{\mu\nu} + iB_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu - \\ & -\frac{i}{2}\int dz d\bar{z}\, \delta_S \left\{ \bar{\psi}^\mu \left(G_{\mu\nu}\dot{X}^\nu - \frac{1}{2}G_{\mu\nu}b^\nu + \frac{1}{2}\Gamma_{\mu\sigma\rho}\bar{\psi}^\sigma\psi^\rho \right) \right\}, \end{aligned} \tag{4.7}$$

where $\Gamma_{\mu\sigma\rho} = \frac{1}{2}(\partial_\sigma G_{\mu\rho} + \partial_\rho G_{\mu\sigma} - \partial_\mu G_{\sigma\rho})$ is the Christoffel symbol. Using (4.4), the gauge fixed action can be written as

$$\begin{aligned} I_{gf} = & \int dz d\bar{z}(\omega_{\mu\nu} + iB_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu - \\ & -\frac{i}{2}\int dz d\bar{z}\Big\{ -\frac{1}{2}G_{\mu\nu}b^\mu b^\nu + b^\mu(G_{\mu\nu}\dot{X}^\nu + \Gamma_{\mu\sigma\rho}\bar{\psi}^\sigma\psi^\rho) - \\ & \qquad - \bar{\psi}^\mu(G_{\mu\nu}\dot{\psi}^\nu + \partial_\rho G_{\mu\nu}\dot{X}^\nu\psi^\rho) + \frac{1}{2}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi^\tau\partial_\tau\Gamma_{\mu\sigma\rho} \Big\}. \end{aligned} \tag{4.8}$$

We now eliminate the auxiliary field b^μ using its equation of motion

$$b^\mu = \dot{X}^\mu + \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho \tag{4.9}$$

and obtain

$$\begin{aligned} I_{gf} = -i\int dz d\bar{z}\Big\{ & (G_{\mu\nu} - B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu - \frac{1}{2}G_{\mu\nu}\bar{\psi}^\mu\dot{\psi}^\nu - \\ & - \frac{1}{2}\Gamma_{\mu\sigma\rho}\bar{\psi}^\mu\dot{X}^\sigma\psi^\rho - \frac{1}{8}R_{\mu\sigma\rho\tau}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi^\tau \Big\}. \end{aligned} \tag{4.10}$$

This corresponds precisely to Witten's topological sigma model[5] (with a B-field included).[6]

4.2 Coupling to topological gravity

Topological string theory is obtained by coupling the topological sigma model to topological gravity [22, 12]; for a review see [25]. The total BRST charge is the sum of two terms,

$$Q_{BRST} = Q_S + Q_V\,, \qquad Q_S^2 = Q_V^2 = \{Q_S, Q_V\} = 0\,. \tag{4.11}$$

Here Q_V corresponds to the "usual" BRST charge Q_B we constructed in section 3 for the bosonic string, while Q_S corresponds to the charge associated with the symmetry δ_S of the topological sigma model.

[5] Our variables $\psi, \bar{\psi}$ and X are identified with the variables $i\chi, -2\psi$ and ϕ of [24], respectively.

[6] We note that the gauge-fixing part of the action can be rewritten as $\{Q_S^+, [Q_S^-, W]\}$ with $W \sim \int d^2\sigma G_{i\bar{j}}\bar{\psi}^i\bar{\psi}^{\bar{j}}$, where Q_S^+ and Q_S^- correspond to the α and $\tilde{\alpha}$ part of the transformations in (3.1) of [24], respectively. Notice that $Q_S = Q_S^+ + Q_S^-$.

As in section 3, the action of Q_V includes

$$\begin{aligned}
\delta_V g_{ab} &= 2C_\omega g_{ab} + \nabla_a c_b + \nabla_b c_a\,, \\
\delta_V \tau^k &= \xi^k\,, \\
\delta_V \tilde{b}^{ab} &= \pi^{ab}\,.
\end{aligned} \tag{4.12}$$

We already know the action of Q_S on the fields of the topological sigma model. Its action on the fields in the gravitational sector is defined by introducing superpartners for the fields in the gravitational sector, including the two-dimensional metric g_{ab}, the moduli τ^k, the auxiliary fields ξ^k and π^{ab}, and the antighost field $\tilde{b}^{ab}$:

$$\begin{aligned}
\delta_S g_{ab} &= \psi_{ab}\,, \\
\delta_S \tau^k &= \hat{\tau}^k\,, \\
\delta_S \xi^k &= \hat{\xi}^k\,, \\
\delta_S p^{ab} &= \pi^{ab}\,, \\
\delta_S \beta^{ab} &= \tilde{b}^{ab}\,.
\end{aligned} \tag{4.13}$$

The ghost fields C_ω and c^a are invariant under Q_S. The Q_V transformation rules of the new fields follow from the anticommutation relations between Q_V and Q_S,

$$\begin{aligned}
\delta_V \psi_{ab} &= (c^c \partial_c \psi_{ab} + \partial_a c^c \psi_{cb} + \partial_b c^c \psi_{ac} + 2C_\omega \psi_{ab})\,, \\
\delta_V \hat{\tau}^k &= -\hat{\xi}^k\,, \\
\delta_V \beta^{ab} &= -p^{ab}\,.
\end{aligned} \tag{4.14}$$

The topological string worldsheet action is the sum of a gravitational term and a sigma model term,

$$L = L_{grav} + L_\sigma. \tag{4.15}$$

The gravitational term is given by

$$\begin{aligned}
L_{grav} &= \frac{1}{4\pi}\delta_V \delta_S \left\{\beta^{ab}[g_{ab} - \hat{g}_{ab}(\tau)]\right\} \\
&= \frac{1}{4\pi}\delta_V \left\{\tilde{b}^{ab}[g_{ab} - \hat{g}_{ab}(\tau)] + \beta^{ab}[\psi_{ab} - \hat{\tau}^k \partial_k \hat{g}_{ab}(\tau)]\right\}.
\end{aligned} \tag{4.16}$$

and is manifestly invariant under both Q_V and Q_S. The sigma model term is manifestly invariant under Q_S. To ensure invariance under Q_V we only need to covariantize the gauge fixing fermion in (4.7). To do this, we note that $\bar{\psi}^i$ is a $(0,1)$ form on M with values in $X^*(T^{1,0})$ and $\bar{\psi}^{\bar{i}}$ is a $(1,0)$ form on M with values in $X^*(T^{0,1})$; making explicit the worldsheet index we have $\bar{\psi}^i_{\bar{z}}$ and $\bar{\psi}^{\bar{i}}_z$, respectively. In other words, the worldsheet holomorphic index is correlated with target space antiholomorphic index and vice versa. This constraint can be imposed covariantly using the projection operator

$$P^{\mu b}_{\nu a} = \frac{1}{2}(\delta^\mu_\nu \delta^b_a - j_a{}^b J^\mu{}_\nu), \tag{4.17}$$

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where a, b are worldsheet indices and $j_a{}^b$ is the worldsheet complex structure (given by[7] $j_a{}^b = -\sqrt{g}\epsilon_{ac}g^{cb}$, $\epsilon_{12} = 1$). We now define

$$\bar{\psi}'^{\mu}_{a} = P^{\mu b}_{\nu a}\bar{\psi}^{\nu}_{b}\,, \qquad b'^{\mu}_{a} = P^{\mu b}_{\nu a}b^{\nu}_{b}\,, \qquad \dot{X}^{\mu}_{a} = 2P^{\mu b}_{\nu a}\partial_b X^{\nu}\,, \tag{4.18}$$

in terms of which the sigma model part of the action is given by

$$\begin{aligned}\int L_\sigma = & \int (\omega_{\mu\nu} + iB_{\mu\nu})dX^\mu \wedge dX^\nu - \\ & -\frac{i}{2}\int d^2\sigma\,\delta_S\left\{\sqrt{g}g^{ab}\bar{\psi}'^{\mu}_{a}(G_{\mu\nu}\dot{X}^{\nu}_{b} - \frac{1}{2}G_{\mu\nu}b'^{\nu}_{b} + \frac{1}{2}\Gamma_{\mu\sigma\rho}\bar{\psi}'^{\sigma}_{b}\psi^{\rho})\right\}.\end{aligned} \tag{4.19}$$

When we work out the action of δ_S in (4.19), we find two contributions. The first comes from δ_S acting on $\sqrt{g}g^{ab}$ and on the projection operators; it has the form

$$\psi_{ab}\mathcal{G}^{ab}_{\sigma}\,, \tag{4.20}$$

where $\mathcal{G}_\sigma$ is the supercurrent of the sigma model. The second contribution comes from δ_S acting on the sigma model fields; it is the familiar sigma model gauge fixing action.

Let us now discuss the insertions in the path integral measure. The terms resulting from the $\tilde{b}$-dependent term in (4.16) are the same as in the bosonic string, so the analysis of the previous section applies. These lead to the usual $(\tilde{b}, \partial_k\hat{g})$ insertions in the path integral. As for the remaining terms, integrating out ψ_{ab} sets the auxiliary field p^{ab} equal to the total supercurrent,

$$p^{ab} = \mathcal{G}^{ab}\,. \tag{4.21}$$

Integrating out $\hat{\tau}^k$ then leads to supercurrent insertions $(\mathcal{G}, \partial_k\hat{g})$. Finally, integrating out $\hat{\xi}^k$ leads to insertions $\delta((\beta, \partial_k\hat{g}))$.

4.3 Gauge dependence and the holomorphic anomaly

We would now like to see what happens if we change the gauge for the sigma model part of the action. To that end, we consider a change of the target space metric, keeping the first line of (4.19) fixed. This clearly changes the gauge fixing condition (as follows from the second line in (4.19)). In terms of complexified Kähler moduli, this corresponds to varying the action with respect to the anti-holomorphic Kähler moduli. To see this, expand $\omega_{\mu\nu}$ and $B_{\mu\nu}$ in a basis of harmonic two-forms $\Omega_{I\mu\nu}$,

$$\omega_{\mu\nu} = \omega^I\Omega_{I\mu\nu}\,, \qquad B_{\mu\nu} = B^I\Omega_{I\mu\nu}\,, \tag{4.22}$$

and define the complexified Kähler moduli

$$t^I = \omega^I + iB^I\,, \qquad \bar{t}^I = \omega^I - iB^I\,. \tag{4.23}$$

The original action (4.1) then depends only on the holomorphic moduli t^I, while the gauge

[7] With these conventions, $j^z{}_z = i$ and $j^{\bar{z}}{}_{\bar{z}} = -i$.

fixing term in (4.7) depends on the combination $t^I + \bar{t}^I$. Varying the metric in the gauge fixing term of the action while keeping the original action fixed thus corresponds to antiholomorphic deformations.

It follows that antiholomorphic dependence of correlation functions is linked to gauge dependence. In the sigma model there is no such dependence since the usual argument leading to (3.30) holds. When coupling to topological gravity, however, the correct Ward identity is (3.33). The boundary contributions in (3.33) have been computed in [19] and do not vanish.[8] Thus we conclude that the theory is gauge dependent due to the holomorphic anomaly.

From the target space point of view, gauge independence of the path integral (with respect to these specific deformations of the gauge fixing condition) would have the interpretation of background independence. The gauge dependence due to holomorphic anomaly has been argued to be a manifestation of a quantum version of background independence [20] (see also [21]). Given that the anomaly originates from boundary terms, it would seem attractive to try and cancel it via a Fischler-Susskind mechanism. In such a scenario the Fischler-Susskind vertex operators would shift the background and this could perhaps realize explicitly the quantum version of background independence. Unfortunately, we have been unable to find appropriate FS vertex operators.

The fact that certain BRST exact states do not decouple due to boundary contributions indicates that that there are degrees of freedom localized at the boundary of moduli space; these are the would-be gauge degrees of freedom that cannot be gauged away because of the anomaly. It would be interesting to understand the physics associated with these degrees of freedom. Similar issues arise when one formulates a QFT on a spacetime with boundaries. To state two examples, a Chern-Simons theory on a 3-manifold with a boundary is not gauge invariant (due to the boundary) and induces a WZW model at the boundary [26]. The second example is AdS gravity: the bulk diffeomorphisms that induce Weyl transformations on the conformal boundary are broken by the holographic Weyl anomaly [27]. In this case the trace of the boundary metric cannot be gauged away and in the AdS/CFT correspondence acts as a source for the trace of the boundary stress energy tensor. The anomalous dependence of the theory on a chosen representative of the boundary conformal structure is readily captured by the anomalous Ward identity similarly to the way the antiholomorphic dependence is captured by the holomorphic anomaly.

5. Conclusions

We have revisited in this paper the BRST quantization of strings. The main difference with previous treatments is that we use the Batalin-Vilkovisky formalism to quantize the worldsheet theory. This treatment automatically incorporates the effects of zero modes. The extraghosts of the BV formalism are identified with the moduli and a measure in moduli space (ghost insertions) uniquely follows from this procedure upon integrating out auxiliary fields. The gauge-fixed action including the ghost insertions is BRST invariant.

[8] To adapt the BRST Ward identity (3.33) to the situation in [19] one should covariantize the sigma model part as given in footnote 6 and extend Q_S^+ and Q_S^- to the gravitational sector.

The BRST transformations however also act on moduli. BRST Ward identities are easily derived (by adapting the QFT derivation of Ward identities) and they incorporate terms due to contributions from the boundary of moduli space (anomalies). These terms arise from the fact that BRST transformation act by a shift on moduli, so they do not leave the integration domain of moduli invariant. We have explicitly discussed bosonic as well as topological strings. In the latter example, the BRST Ward identities give rise to the holomorphic anomaly.

The procedure discussed here is very efficient in determining the path integral measure. We should note however that additional steps may be required when several patches are needed in order to cover the moduli space. In particular, one would need to pass from local to global data. This may be done by utilizing the Čech-De Rham cohomology, as is discussed in the context of topological gravity in [14].[9]

It would be interesting to apply the method discussed here to quantize the Ramond-Neveu-Schwarz (RNS) superstring. Note that a naive integration over moduli space leads to gauge dependent results at two loops [30]. The measure at genus 2 was recently constructed in a series of paper by D'Hoker and Phong, see [29] for a review. It would also be interesting to apply our formalism to derive the measure of superstring in the pure spinor formalism [31]. In this respect, we note that it is straightforward [32] to use the methods here to derive the path integral measure for a related model [33].

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References

[1] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*, Cambridge University Press, Cambridge 1998.

[2] E. D'Hoker, *String theory*, in *Quantum fields and strings: a course for mathematicians*, vol. 2, p. 807. P. Deligne et al. eds., AMS, Providence 1999.

[3] E. D'Hoker and D.H. Phong, *The geometry of string perturbation theory*, *Rev. Mod. Phys.* **60** (1988) 917.

[4] D. Friedan, *Introduction to Polyakov's string theory*, in *Les Houches summer school, recent advances in field theory and statistical mechanics*, Z. Zuber and R. Stora eds., North-Holland, 1984.

[5] O. Alvarez, *Theory of strings with boundaries: fluctuations, topology and quantum geometry*, *Nucl. Phys.* **B 216** (1983) 125.

[9] The relevance of the Čech-De Rham cohomology for superstrings is discussed in [28] (as cited in [29, 14]).

[6] G.W. Moore and P. Nelson, *Absence of nonlocal anomalies in the Polyakov string*, *Nucl. Phys.* **B 266** (1986) 58.

[7] E. D'Hoker and D.H. Phong, *Multiloop amplitudes for the bosonic Polyakov string*, *Nucl. Phys.* **B 269** (1986) 205.

[8] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[9] I.A. Batalin and G.A. Vilkovisky, *Quantization of gauge theories with linearly dependent generators*, *Phys. Rev.* **D 28** (1983) 2567, erratum *ibid.* **D 30** (1984) 508.

[10] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, Princeton 1992.

[11] L. Baulieu and M. Bellon, *BRST symmetry for finite dimensional invariances: applications to global zero modes in string theory*, *Phys. Lett.* **B 202** (1988) 67.

[12] J.M.F. Labastida, M. Pernici and E. Witten, *Topological gravity in two-dimensions*, *Nucl. Phys.* **B 310** (1988) 611.

[13] P. Mansfield, *The consistency of topological expansions in field theory: 'BRST anomalies' in strings and Yang-Mills*, *Nucl. Phys.* **B 416** (1994) 205 [hep-th/9308117].

[14] C.M. Becchi and C. Imbimbo, *Gribov horizon, contact terms and Čech-De Rham cohomology in 2d topological gravity*, *Nucl. Phys.* **B 462** (1996) 571 [hep-th/9510003].

[15] E. Cohen, C. Gomez and P. Mansfield, *BRS invariance of the interacting Polyakov string*, *Phys. Lett.* **B 174** (1986) 159.

[16] P. Mansfield, *Nilpotent brst invariance of the interacting Polyakov string*, *Nucl. Phys.* **B 283** (1987) 551.

[17] C.G. Callan Jr., C. Lovelace, C.R. Nappi and S.A. Yost, *Adding holes and crosscaps to the superstring*, *Nucl. Phys.* **B 293** (1987) 83.

[18] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, *Nucl. Phys.* **B 296** (1988) 91.

[19] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[20] E. Witten, *Quantum background independence in string theory*, hep-th/9306122.

[21] R. Dijkgraaf, E. Verlinde and M. Vonk, *On the partition sum of the NS five-brane*, hep-th/0205281.

[22] E. Witten, *Topological sigma models*, *Commun. Math. Phys.* **118** (1988) 411.

[23] L. Baulieu and I.M. Singer, *The topological sigma model*, *Commun. Math. Phys.* **125** (1989) 227.

[24] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[25] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Notes on topological string theory and 2-d quantum gravity*, PUPT-1217, based on lectures given at *Spring school on strings and quantum gravity*, Trieste, Italy, Apr 24 - May 2, 1990 and at *Cargese workshop on random surfaces, quantum gravity and strings*, Cargese, France, May 28 - Jun 1, 1990.

[26] S. Elitzur, G.W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, *Nucl. Phys.* **B 326** (1989) 108.

[27] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087].

[28] H. Verlinde, *A note on the integral over fermionic supermoduli*, Utrecht preprint No. THU-87/26 (1987), unpublished.

[29] E. D'Hoker and D.H. Phong, *Lectures on two-loop superstrings*, hep-th/0211111.

[30] E. Verlinde and H. Verlinde, *Multiloop calculations in covariant superstring theory*, *Phys. Lett.* **B 192** (1987) 95.

[31] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047 [hep-th/0406055].

[32] P.A. Grassi and K. Skenderis, unpublished.

[33] P.A. Grassi and G. Policastro, *Super-Chern-Simons theory as superstring theory*, hep-th/0412272.